

BOUNDARY CONTINUITY OF COMPLETE PROPER HOLOMORPHIC MAPS

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Abstract We show that there is no complete proper holomorphic map from the disc Δ to the bidisc Δ^2 which extends continuously through $\overline{\Delta}$.

Let Δ be the open unit disc in \mathbb{C} and let $N \geq 2$. A holomorphic immersion $\varphi: \Delta \rightarrow \mathbb{C}^N$ is called *complete* if the pullback of the Euclidean metric φ^*g is a complete metric on Δ . This is equivalent to saying that for every path $p: [0, 1) \rightarrow \Delta$ such that $p(t) \rightarrow b\Delta$ as $t \rightarrow 1$, the composition $\varphi \circ p$ has infinite Euclidean length. Notice that this equivalent statement makes sense as the definition of completeness for general holomorphic maps.

Answering a question of P. Yang [Y], P. W. Jones [J] was the first to show that there are bounded complete holomorphic immersions. It is now known that given a convex domain $D \subset \mathbb{C}^2$ there is a complete proper holomorphic immersion $\varphi: \Delta \rightarrow D$ [AL]. In [AF] this was generalized to the case where Δ is replaced with a bordered Riemann surface.

It is a natural question whether, in the case of bounded D , there is such a φ which extends continuously through $\overline{\Delta}$. For instance, if D is a ball, does there exist a complete, proper holomorphic map $\varphi: \Delta \rightarrow D$ which extends continuously through $\overline{\Delta}$? This is an open question. In the present note we show that for general bounded convex domains D the answer is no:

PROPOSITION *Let $F = (f, g): \Delta \rightarrow \Delta^2$ be a complete, proper holomorphic map. There is no arc $\Lambda \subset b\Delta$ such that $z \mapsto |f(z)|$ extends continuously to $\Delta \cup \Lambda$.*

Proof. Suppose that $\Lambda \subset b\Delta$ is an open arc such that $z \mapsto |f(z)|$ extends continuously to $\Delta \cup \Lambda$. Suppose first that the continuous extension of $z \mapsto |f(z)|$ is identically equal to 1 on Λ . By the Schwarz reflection principle [R, p. 237; p. 293, Ex.2] f extends holomorphically across Λ . Thus, for every $e^{i\theta} \in \Lambda$ we have

$$\int_0^1 |f'(te^{i\theta})| dt < \infty. \quad (1)$$

Since the function g is bounded, a result of J. Bourgain [B] implies that there is an $e^{i\alpha} \in \Lambda$ such that

$$\int_0^1 |g'(te^{i\alpha})| dt < \infty.$$

By (1) it follows that

$$\int_0^1 \sqrt{|f'(te^{i\alpha})|^2 + |g'(te^{i\alpha})|^2} dt < \infty$$

which contradicts the completeness of F .

Thus, there is a point $w \in \Lambda$ such that the extension of $z \mapsto |f(z)|$ at w is less than one. By the continuous extendibility of $z \mapsto |f(z)|$ to $\Delta \cup \Lambda$ there are a closed arc $A \subset \Lambda$, a neighbourhood \mathcal{W} of A in \mathbb{C} and an $\eta > 0$ such that

$$|f(z)| < 1 - \eta \quad (z \in \mathcal{W} \cap \Delta).$$

Since the map F is proper it follows that if $z \in A$ and $\zeta \in \Delta$, $\zeta \rightarrow z$ then $|g(\zeta)| \rightarrow 1$. This means that $z \mapsto |g(z)|$ extends continuously to $\Delta \cup A$ and that the extension is identically equal to 1 on A . This is impossible by the first part of the proof with the roles of f and g interchanged. This completes the proof.

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